# How weak is weak extent?

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**Abstract.** We show that the extent of a Tychonoff space of countable weak extent can be arbitrary big. The extent of $X$ is $e(X) = \sup\{|F| : F \subset X$ is closed and discrete$\}$ while $we(X) = \min\{\tau :$ for every open cover $\mathcal{U}$ of $X$ there is $A \subset X$ such that $|A| \leq \tau$ and $St(A,\mathcal{U}) = X\}$ is the weak extent of $X$ (also called the star-Lindelöf number of $X$). Also we show that the extent of a normal space with countable weak extent is not greater than $\mathbf{c}$.

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Recall that the extent of a topological space $X$ is the cardinal $e(X) = \sup\{|F| : F \subset X$ is closed and discrete$\}$. The *weak extent* of $X$ is the cardinal $we(X) = \min\{\tau :$ for every open cover $\mathcal{U}$ of $X$ there is $A \subset X$ such that $|A| \leq \tau$ and $St(A,\mathcal{U}) = X\}$ [5]. The reason for this name is that for any $X \in \mathrm{T}_1$, $we(X) \leq e(X)$; indeed, supposing $we(X) > \kappa$, there is an open cover such that for every $A \subset X$ with $|A| \leq \kappa$ one has $St(A,\mathcal{U}) \neq X$; then one can inductively choose points $x_\alpha$, $\alpha < \kappa$, so that $x_\alpha \notin St(\{x_\beta : \beta < \alpha\},\mathcal{U})$ for each $\alpha$; once the points have been choosen the set $\{x_\alpha : \alpha < \kappa\}$ is closed, discrete and of cardinality $\kappa$, so $e(X) \geq \kappa$. Note also that $we(X) \leq d(X)$ obviously holds for every $X$. Some cardinal inequalities involving extent can be improved by replacing extent by weak extent. Thus for $X \in T_1$, $|K(X)| \leq we(X)^{psw(X)}$ [5]. A natural question was stated in [1], [9]: how big can the difference between the extent and the weak extent of a $\mathrm{T}_i$ space be? First, we give the answer for the Tychonoff case.

**Theorem 1** *For every cardinal $\tau$ there is a Tychonoff space $X$ such that $e(X) \geq \tau$ and $we(X) = \omega$.*

Before the paper [5] the cardinal function $we(X)$ was called the *star-Lindelöf number* [10], [8], [11]. In particular, a space $X$ such that $we(X) = \omega$ is called star-Lindelöf or *Lindelöf, see e.g. [6], [1], [2] [3].

Note that $e(X) \leq 2^{we(X)\chi(X)}$ for every regular space $X$ [1].

To prove Theorem 1, we use a set-theoretic fact in Theorem 2 below.. Let $S$ be a set and $\lambda$ a cardinal. A *set mapping* of order $\lambda$ on $S$ is a mapping that assigns to each $s \in S$ a subset $f(s) \subset S$ so that $|f(s)| < \lambda$ and $s \notin f(s)$. A subset $T \subset S$ is called *$f$-free* if $f(t) \cap T = \emptyset$ for every $t \in T$. Answering a question of Erdös, Fodor proved in 1952 ([4], see also [12], Theorem 3.1.5) a general theorem a partial case of which is the following

**Theorem 2** (Fodor) *Let $S$ be a set of cardinality $\tau$ and let $f$ be a set mapping on $S$ of order $\omega$. Then there is a countable family $\mathcal{H}$ of $f$-free subsets of $S$ such that $\cup\mathcal{H} = S$.*

**Proof of Theorem 1:** Let $\tau$ be an infinite cardinal. For each $\alpha < \tau$, $z_\alpha$ denotes the point in $D^\tau$ with only the $\alpha$-th coordinate equal to 1. Put $Z = \{z_\alpha : \alpha < \tau\}$. Then $Z$ is a discrete subspace of $D^\tau$. Further, let $\kappa$ be a cardinal such that $\mathrm{cf}(\kappa) > \tau$. Put

$$X = (D^\tau \times (k+1)) \setminus ((D^\tau \setminus Z) \times \{\kappa\}) .$$

Also we denote $X_0 = D^\tau \times \kappa$ and $X_1 = Z \times \{\kappa\} = \{(z_\alpha, \kappa) : \alpha < \tau\}$. Then $X = X_0 \cup X_1$.

It is clear that $X_1$ is closed in $X$ and discrete, so $e(X) \geq \tau$.

It remains to prove that $we(X) = \omega$. First, note that $X_0$ is countably compact, hence star-Lindelöf. So it remains to prove that $X_1$ is relatively star-Lindelöf in $X$, i.e. for every open cover $\mathcal{U}$ of $X$ there is a countable $A \subset X$ such that $St(A, \mathcal{U}\} \supset X_1$.

Let $\mathcal{U}$ be an open cover of $X$. For every $\alpha < \tau$ choose an $U_\alpha \in \mathcal{U}$ so that $(z_\alpha, \kappa) \in U_\alpha$. Further, for every $\alpha < \tau$ choose $\xi_\alpha < \kappa$ and $B_\alpha$, an element of the standard base of $D^\tau$, so that $(z_\alpha, \kappa) \in (B_\alpha \times (\xi_\alpha, \kappa]) \cap X \subset U_\alpha$. It remains to check that

(+) there is a countable $C \subset D^\tau$ such that $B_\alpha \cap C \neq \emptyset$ for every $\alpha < \tau$.

Indeed, since $\mathrm{cf}(\kappa) > \tau$, there is a $\gamma < \kappa$ such that $\gamma > \xi_\alpha$ for all $\alpha < \tau$. Put $A = C \times \{\gamma\}$. Then $U_\alpha \cap A \neq \emptyset$ for all $\alpha < \tau$, so $X_1 \subset St(A, \mathcal{U})$.

Now we check (+). The set $B_\alpha$ has the form

$$B_\alpha = \{x \in D^\tau : x(\alpha) = 1 \text{ and } x(\alpha') = 0 \text{ for all } \alpha' \in A_\alpha\}$$

where $A_\alpha$ is some finite subset of $\tau \setminus \{\alpha\}$. Consider the set mapping $f$ that assigns $A_\alpha$ to $\alpha$ for each $\alpha < \tau$. By Fodor's theorem, there is a countable, $f$-free family $\mathcal{H} = \{H_n : n \in \omega\}$ of subsets of $\tau$ such that $\cup\mathcal{H} = \tau$. For each $n \in \omega$, denote by $c_n$ the indicator function of $H_n$, i.e. $c_n(\alpha) = 1$ iff $\alpha \in H_n$. Since $H_n$ is $f$-free, $B_\alpha \ni c_n$ for all $\alpha \in H_n$. Put $C = \{c_n : n \in \omega\}$. Then $B_\alpha \cap C \neq \emptyset$ for every $\alpha < \tau$, i.e. (+) holds.

Pseudocompactness of $X$ follows from the fact that $X$ contains a dense countably compact subspace $X_0$. $\square$

Now we are going to show that in the normal case the extent of a space of countable weak extent is not greater than $\mathbf{c}$. In fact, we will prove a slightly more general statement. Recall that a family of sets is linked if every two elements have nonempty intersection. The linked-Lindelöf number of $X$ is the cardinal $ll(X) = \min\{\tau :$ every open cover of $X$ has a subcover representable as the union of at most $\tau$ many linked subfamilies$\}$ [3]. A space $X$ with $ll(X) = \omega$ is called linked-Lindelöf [3]. It is easy to see that $ll(X) \leq we(X)$ for every $X$.

**Theorem 3** *For every normal space $X$, $e(X) \leq 2^{ll(X)}$.*

**Proof:** Let $\tau$ be an infinite cardinal, $K$ a closed discrete subspace of a normal space $X$ and $|K| = k > 2^\tau$. We have to show that $ll(X) > \tau$. It is easy to construct a family $\mathcal{A}$ of subsets of $K$ such that $|\mathcal{A}| = k$ and for every nonempty finite subfamily of $\mathcal{A}$, say $A_1, \ldots, A_n, A_{n+1}, \ldots, A_{n+m}$,

$$(*) \quad |A_1 \cap \ldots \cap A_n \cap (K \setminus A_{n+1}) \cap \ldots \cap (K \setminus A_{n+m})| = k.$$

For every $A \in \mathcal{A}$ pick a continuous function $f_A : X \to I$ such that $f(A) = \{1\}$ and $f(K \setminus A) = \{0\}$. Denote $\mathcal{F} = \{f_A : A \in \mathcal{A}\}$ and $F = \Delta\mathcal{F} : X \to I^{\mathcal{F}}$. Then $|\mathcal{F}| = k$. Note that $F(K) \subset D^{\mathcal{F}}$. It follows from $(*)$ that $F(K)$ is dense in $D^{\mathcal{F}}$, moreover, every open set in $D^{\mathcal{F}}$ containes $k$ elements of $F(K)$. There is therefore a bijection $\varphi : K \to \mathcal{B}$, where $\mathcal{B}$ is the standard base of

$D^{\mathcal{F}}$, such that $\varphi(z) \ni F(z)$ for every $z \in K$. Every element $B \in \mathcal{B}$ has the form

$$B = B^{i_1 \dots i_n}_{f_1 \dots f_n} = \{x \in D^{\mathcal{F}} : x(f_1) = i_1, \dots, x(f_n) = i_n\}$$

where $n \in \mathbb{N}$, $f_1, \dots f_n \in \mathcal{F}$ and $i_1, \dots, i_n \in D$. Denote

$$U(B) = \left\{ x \in I^{\mathcal{F}} : \forall j \in \{1, \dots, n\} \begin{pmatrix} x(f_j) > \frac{1}{2} \text{ if } i_j = 1 \\ x(f_j) < \frac{1}{2} \text{ if } i_j = 0 \end{pmatrix} \right\}.$$

Further, for every $z \in K$ put $\tilde{\varphi}(z) = U(\varphi^{-1}(z))$. Then $\tilde{\varphi}(z)$ is a neighbourhood of $F(z)$ in $I^{\mathcal{F}}$. Note that

$$(**) \quad \tilde{\varphi}(z) \cap \tilde{\varphi}(z') \neq \emptyset \quad \text{iff} \quad \varphi(z) \cap \varphi(z') \neq \emptyset$$

Let $\mathcal{G}$ denote the family of all continuous functions form $X$ to $I$, $G = \Delta \mathcal{G} : X \to I^{\mathcal{G}}$, $\pi : I^{\mathcal{G}} \to I^{\mathcal{F}}$ is the natural projection. For each $z \in K$ denote $\tilde{\tilde{\varphi}}(z) = \pi^{-1}(\varphi(z))$. Then $\tilde{\tilde{\varphi}}(z)$ is a neighbourhood of $G(z)$ in $I^{\mathcal{G}}$ and

$$(***) \quad \tilde{\tilde{\varphi}}(z) \cap \tilde{\tilde{\varphi}}(z) \neq \emptyset \quad \text{iff} \quad \tilde{\varphi}(z) \cap \tilde{\varphi}(z') \neq \emptyset.$$

Last, for every $z \in K$ put $\tilde{\tilde{\tilde{\varphi}}}(z) = (\tilde{\tilde{\varphi}}(z) \setminus G(K \setminus \{z\})) \cap G(X)$. Then $\tilde{\tilde{\tilde{\varphi}}}(z)$ is a neighbourhood of $G(z)$ in $G(X)$ and

$$(*\text{v}) \quad \tilde{\tilde{\tilde{\varphi}}}(z) \cap \tilde{\tilde{\tilde{\varphi}}}(z') \neq \emptyset \quad \text{iff} \quad \tilde{\tilde{\varphi}}(z) \cap \tilde{\tilde{\varphi}}(z) \neq \emptyset.$$

Put $\mathcal{U}_0 = \{\tilde{\tilde{\varphi}}(z) : z \in K\}$. Since $G$ is a homeomorphic embedding, $G(K)$ is closed in $G(X)$, so $O = G(X) \setminus G(K)$ is open and hence $\mathcal{U} = \mathcal{U}_0 \cup \{O\}$ is an open cover of $G(X)$.

Since $w(D^{\mathcal{F}}) > 2^\tau$, $\mathcal{B}$, a base of $D^{\mathcal{F}}$, is not representable as the union of at most $\tau$ many linked subfamilies (see e.g. [7]). By $(**)$, $(***)$ and $(*\text{v})$ the same can be said about the family $\mathcal{U}_0$. Note that for every $z \in K$, $\tilde{\tilde{\tilde{\varphi}}}(z)$ is the only element of $\mathcal{U}$ that containes $z$. So $\mathcal{U}$ does not have a subcover representable as the union of at most $\tau$ many linked subfamilies and thus $ll(X) = ll(G(X)) > \tau$.$\Box$

It is not clear whether the inequality in the previous theorem can be made strict, even with star-Lindelöf number instead of linked-Lindelöf.

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